Notes for lectures 04-05

Looking in detail at the parametrization, for $n = 2$

Extending the pieces of $F(\lambda)$

1 *What we have so far.* In the last two classes, we have proven that MaxFlow has high parametric complexity by:

1. producing a parametrization P_n, an interval of definition $I = [-T, T]$, and 2^n intervals $I(\sigma), \sigma \in \{-1, +1\}^n$, such that
2. for all $\lambda \in I(\sigma)$ the optimum solution is fixed, and the optimum value $f(P_n(\lambda))$ is a linear function $A(\sigma)\lambda + B(\sigma)$, with
3. $|A(\sigma) - A(\sigma')| \geq 1$ when $\sigma \neq \sigma'$, [1] and
4. $I(\sigma)$ has length at least 1 [2].

Now notice that if the optimization problem is homogeneous[3], then we can take any such parametrization P_n, and produce a parametrization P'_n of the same complexity, but where $I(\sigma)$ now has length at least 2^n. To do that, replace any parameterized numerical input $\alpha\lambda+\beta$ in P_n with $\alpha\lambda+2^n\beta$ in P'_n, while also extending the interval of definition to $[-2^nT, 2^nT]$ — i.e., we scale λ and the whole graph $F(\lambda)$ by a 2^n factor. This has increased the bit-length of P_n by at most n bits, so the bit-length of P'_n it is still $O(n^2)$. Also, conditions 2 and 3 above still hold.

The lower bound

2 *Bound on the number of polygons in arrangements of real lines.* Any n lines divide $\mathbb{R}^2$ into at most $1 + n + \binom{n}{2} \leq n^2$ regions.

[1] Recall that $A(\sigma) = \sum_{i\in[n]} \sigma_i w_i$.

[2] In fact it had length exactly 2, since we break length $|I| = 2^{n+2}$ into 2^n intervals of length 4, and remove length 1 from both sides

[3] i.e., extendable to reals with $f(\alpha\bar{w}) = \alpha f(\bar{w})$ for any $\alpha > 0$

Proof. If $n-1$ lines divide $\mathbb{R}^2$ into p regions, then by adding one more line, some extra regions are produced. How many? Well, among the p existing regions, the new line may cross some of them, and maybe not others. The new line divides each region it crosses into two.

Whenever the new line changes the region it is crossing, it must have passed through one of the old lines, which it does only once for each old line, and there are $n-1$ many old lines; hence the new line can cross at most $1+n-1$ regions, and hence all of the n lines divide $\mathbb{R}^2$ into at most $p+n$ regions. And $2+2+3+\ldots+n = 1+n+\binom{n}{2} \leq n^2$. ∎

3 Theorem (branching bound). Suppose that the language $\mathcal{A} \subseteq \mathsf{Dom}(\mathcal{A}) \subseteq \mathbb{N}^2$ is decided in linKC$(p(n), t(n))$. Then $\mathbb{R}^2$ can be divided by a set of at most

$$p(n)^{2t(n)} p(n) t(n)$$

lines, such that each polygon in the resulting arrangement can be labeled *yes* or *no* in a way such that a given $(\lambda, z) \in \mathsf{Dom}(\mathcal{A})$ is in $\mathcal{A}$ if and only if it lies in a polygon labeled *yes*.

Proof. Suppose that such a KC-algorithm A existed for deciding $\mathcal{A}$ in $t(n)$-time using $p(n)$ processors.

Call two inputs x and x' in $\mathsf{Dom}(\mathcal{A})$ t-equivalent if for both x and x' every processor of A branched the same way up to time t for both inputs x and x'.

How many t-equivalent-classes are there? For given t-equivalence-class C, the content at time t of each memory register r is given by a linear form in the input $\ell_r(x)$, i.e., a function of the form $\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$. To each branch instruction, e.g., `if` $\ell_r(x) \leq 0$ `goto` p, corresponds such a register r. Consider the arrangement in $\mathbb{R}^2$ formed by lines $\ell_r = 0$ corresponding to all the ℓ_r that appear in some branch instruction at time $t+1$. By §2, this arrangement divides $\mathbb{R}^2$ into at most $p(n)^2$ polygons (which are disjoint except for their shared border). If two inputs $x, x' \in C$ belong to the same polygon, then every branch instruction branches in the same way for x and x' at time $t+1$. Hence, any t-equivalence-class is divided up into at most $p(n)^2$ $(t+1)$-equivalence-classes.

Hence there are at most $p(n)^{2t(n)}$ $t(n)$-equivalence classes. Now let us consider all linear forms giving the contents of each register belonging to some branching instruction at some time-step, over all equivalence classes. There are at most $p(n)^{2t(n)} p(n) t(n)$ such linear forms. Let D be the set of lines $\ell(x) = 0$, for each such linear form; the lines in D divide $\mathbb{R}^2$ into polygons, and if two inputs x, x' belong to the same polygon, the output of A on x and x' must be the same. Hence these polygons may be labeled so that they correctly classify inputs according to $\mathcal{A}$. ■

4 Corollary. Let P_n be our parametrization for MaxFlow, having complexity $\phi \geq 2^n$. Now, suppose that $\mathcal{A} = \{(\lambda, z) | f(P_n(\lambda)) \leq z\}$ is decidable in $\mathrm{linKC}(2^{t(n)}, t(n))$ [4].

Then $\mathbb{R}^2$ can be divided by at most $2^{3t(n)^2}$ real lines, and each polygon of the resulting arrangement can be labeled *yes* or *no*, in such a way that $(\lambda, z) \in I_n \times \mathbb{N}$ is in $\mathcal{A}$ iff it lies in a polygon labeled *yes*.

☞ For a given P_n with $\mathcal{A} \in \mathrm{linKC}(2^{t(n)}, t(n))$, let $D = D(n)$ denote such a set of $d = d(n) = |D(n)| \leq 2^{3t(n)^2}$ lines (whose induced polygons in $\mathbb{R}^2$ correctly classify the λ, z pairs).

5 *Good polygon.* Let G denote the graph of $f(P_n(\lambda))$ for $\lambda \in I_n$; i.e.,

$$G = \{(\lambda, f(P_n(\lambda))) | \lambda \in I_n\}.$$

Recall that G is piecewise linear, and that the number of line-segments of G is the complexity ϕ of P_n. For a given line segment S of G, its *integer points* are those points $(\lambda, f(P_n(\lambda)) \in S$ with an integer λ. Recall that we may assume that each segment of G contains at least 2^n integer points.

The set D of lines induces a set of polygons in $\mathbb{R}^2$, and a black-white colouring for these polygons tha correctly classifies $\mathcal{A}$. We say that a polygon in this set is *good* for a line segment S in G if it correctly classifies at least $\lfloor 2^n/d \rfloor$ integer points of S.

6 Lemma. D induces a polygon that is good for $\lfloor \phi/d \rfloor$ of the line segments in G.

Proof. Any integer point in a given segment S of G is correctly classified by some polygon; because there are at least 2^n such integer points, then by the pidgeonhole principle there must exist some polygon that is good for S. But there are ϕ line segments in G, hence the lemma follows, again by the pidgeonhole principle. ■

7 Lemma. If $d \ll 2^{n-1}$, no polygon with k sides can be good for $k+1$ such line segments.

Proof. The lemma will follow once we show that no single side of a polygon can be good for two segments.

[4] Where f gives the maximum-flow, λ is any integer in the interval of definition I_n of P_n, and z is any integer.

Indeed, if a given side $s = a\lambda + b$ of a polygon is good for a given segment $S = A\lambda + B$, then there are two sample points $p_1 = A\lambda_1 + B$ and $p_2 = A\lambda_2 + B$ in S_1 where λ_1 and λ_2 are at least $2^n/d$ distance apart. But because s is above (λ_i, p_i) and below $(\lambda_i, p_i + 1)$ for both $i = 1, 2$, it now follows that $|a - A| \leq \frac{d}{2^n} \ll 1/2$. Now, because the slopes of each segment in G are at least 1 distance apart, it follows that s can not be good for any other segment in G. ■

8 Corollary. $\mathcal{A} \notin \mathrm{linKC}(2^{\sqrt{n}/3}, \sqrt{n}/3)$.

Proof. If $\mathcal{A} \in \mathrm{linKC}(2^{\sqrt{n}/3}, \sqrt{n}/3)$, then by §4 we could find a set D of $d \leq 2^{\frac{3}{9}n} \ll 2^{n-1}$ real lines, whose resulting arrangement correctly classifies $I_n \times \mathbb{N}$ according to $\mathcal{A}$. But then by §6, some polygon in this arrangement is good for $2^{\frac{6}{9}n}$ line segments of G, and hence by §7, this polygon has at least $2^{\frac{6}{9}n}$ sides... but this is more than the number of lines in D! Hence the theorem is proven by contradiction. ■